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Cubic supersymmetry and abelian gauge invariance

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Abstract

On the basis of recent results extending non-trivially the Poincaré symmetry, we investigate the properties of bosonic multiplets including 2−form gauge fields. Invariant free Lagrangians are explicitly built which involve possibly 3− and 4−form fields. We also study in detail the interplay between this symmetry and a $U(1)$ gauge symmetry, and in particular the implications of the automatic gauge-fixing of the latter associated to a residual gauge invariance, as well as the absence of self-interaction terms.

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1 Introduction

A non-trivial extension of the Poincaré algebra, different from the supersymmetric one, was introduced in [1]. The main idea is to consider an adapted algebraic structure, named $F-$Lie algebra, which is a generalisation of Lie superalgebras. Consequently, from the very beginning this construction evades the no-go theorem of Haag-Lopuszanski-Sohnius [2]. Similarly to Lie superalgebras which underly the structure of supersymmetry, $F-$Lie algebras underly that of fractional supersymmetry [3].

A specific $F-$Lie algebra (for $F = 3$) has been studied and leads to a quantum field theoretical realisation of a non-interacting theory, named *cubic supersymmetry* or 3SUSY [4]. In this new algebraic frame, one does not consider square roots of translations ($QQ \sim P$), as it is the case for supersymmetry, but rather cubic roots ($QQQ \sim P$). The representation theory of 3SUSY has been investigated and leads either to pure bosonic or to pure fermionic multiplets. The situation is drastically different from supersymmetry, since the multiplets contain only states with the same statistics. This is due to the fact that in our algebra, the additional generators Q belong to the vector representation of the Poincaré algebra, while in the SUSY case the additional generators belong to the spinorial representation of the Poincaré algebra.

In this paper we investigate the properties of the bosonic multiplets which involve scalar, vectors and $2-$forms. In the section 2 we firstly recall some basic results already obtained in [4]. Then, we explicitly diagonalise the Lagrangian obtained in [4]. We observe that 3SUSY invariance requires gauge fixing terms *à la* Feynman, for the vectors and the $2-$forms. This Lagrangian has wrong signs in the kinetic term of some of the fields, thus leading to unboundedness from below for these energy densities. We propose here a possible solution to this problem. Indeed, using the Hodge duality for the $p-$forms and the specific form of the Lagrangian (kinetic term + gauge fixing term), the physical field is interpreted as $^\star A$, the Hodge dual of A, instead of A. This mechanism leads to $3-$ and $4-$forms. Then, quadratic couplings between different types of bosonic mutiplets are taken into account. The total free Lagrangian is then diagonalised, leading to (i) constraints on the coupling parameters in order to have positive square mass, and (ii) non-conventional kinetic terms for the $2-$forms. Section 3 is devoted to the proof that no 3SUSY invariant interacting terms are possible within these bosonic multiplets. In this section we also recall some relations satisfied by the (anti)-self-dual 2-froms, and establish a useful property for the derivatives of the various multiplets. In section 4, we study the compatibility between the 3SUSY and $U(1)$ (gauge) symmetries, point out the existence of an induced symmetry, and determine explicitly the functional subclass of the allowed gauge transformations. We also comment briefly on tentative superspace formulation. Section 5 contains the conclusions and some perspectives as regards the interaction possibilities.

2 Free theory

2.1 Algebra and self-coupling of multiplets

The 3SUSY algebra is constructed from the Poincaré generators P_m, L_{mn} with additional generators Q_m in the vector representation of the Lorentz group [4]

$$
\begin{aligned}
&[L_{mn}, L_{pq}] = \eta_{nq} L_{pm} - \eta_{mq} L_{pn} + \eta_{np} L_{mq} - \eta_{mp} L_{nq},\ [L_{mn}, P_p] = \eta_{np} P_m - \eta_{mp} P_n, \\
&[L_{mn}, Q_p] = \eta_{np} Q_m - \eta_{mp} Q_n, \quad [P_m, Q_n] = 0, \\
&\{Q_m, Q_n, Q_r\} = \eta_{mn} P_r + \eta_{mr} P_n + \eta_{rn} P_m,
\end{aligned} \tag{2.1}
$$

where $\{Q_m, Q_n, Q_p\} = Q_m Q_n Q_r + Q_m Q_r Q_n + Q_n Q_m Q_r + Q_n Q_r Q_m + Q_r Q_m Q_n + Q_r Q_n Q_m$ stands for the symmetric product of order 3 and $\eta_{mn} = \text{diag}\,(1, -1, -1, -1)$ is the Minkowski metric. Two irreducible matrix representations have been found

$$
Q_{+m} = \begin{pmatrix} 0 & \Lambda^{1/3}\sigma_m & 0 \\ 0 & 0 & \Lambda^{1/3}\bar{\sigma}_m \\ \Lambda^{-2/3} P_m & 0 & 0 \end{pmatrix}, Q_{-m} = \begin{pmatrix} 0 & \Lambda^{1/3}\bar{\sigma}_m & 0 \\ 0 & 0 & \Lambda^{1/3}\sigma_m \\ \Lambda^{-2/3} P_m & 0 & 0 \end{pmatrix} \tag{2.2}
$$

with $\sigma^m = (\sigma^0 = 1, \sigma^i)$, and $\bar{\sigma}^m = (\bar{\sigma}^0 = 1, -\sigma^i)$, σ^i the Pauli matrices and Λ a parameter with mass dimension that we take equal to 1 (in appropriate units). These matrix representations give rise to fermionic and bosonic multiplets [4]. Here, we just consider the following bosonic multiplets [4]:

$$
\begin{aligned}
\Xi_{++} &= \begin{pmatrix} \varphi, B_{mn} \\ \tilde{A}_m \\ \tilde{\tilde{\varphi}}, \tilde{\tilde{B}}_{mn} \end{pmatrix} \qquad & \Xi_{+-} &= \begin{pmatrix} A'_m \\ \tilde{\varphi}', \tilde{B}'_{mn} \\ \tilde{\tilde{A}}'_m \end{pmatrix} \\
\Xi_{--} &= \begin{pmatrix} \varphi', B'_{mn} \\ \tilde{A}'_m \\ \tilde{\tilde{\varphi}}', \tilde{\tilde{B}}'_{mn} \end{pmatrix} \qquad & \Xi_{-+} &= \begin{pmatrix} A_m \\ \tilde{\varphi}, \tilde{B}_{mn} \\ \tilde{\tilde{A}}_m \end{pmatrix}
\end{aligned} \tag{2.3}
$$

where $\varphi, \tilde{\tilde{\varphi}}, \varphi', \tilde{\tilde{\varphi}}', \tilde{\varphi}, \tilde{\varphi}'$ are scalars fields, $\tilde{A}, \tilde{A}', A, \tilde{\tilde{A}}, A', \tilde{\tilde{A}}'$ are vector fields, $B, \tilde{B}, \tilde{\tilde{B}}$ are self-dual $2-$forms and $B', \tilde{B}', \tilde{\tilde{B}}'$ are anti-self-dual $2-$forms. The 3SUSY algebra (2.1) and its representations (2.2) are $\mathbb{Z}_3-$graded. Therefore, one can assume that, for example, for the multiplet Ξ_{++}, the fields φ, B are in the $(-1)-$graded sector, $\tilde{A}$ in the $0-$graded sector and $\tilde{\tilde{\varphi}}, \tilde{\tilde{B}}$ in the $1-$graded sector. The same classification also holds for the other multiplets. Furthermore, due to the property of (anti-)self-duality of $2-$forms in $4D$, ${}^\star B = iB, {}^\star B' = -iB'$, *etc* (with ${}^\star B$ the Hodge dual of B), the $2-$forms are complex representations of $\mathfrak{so}(1,3)$ and consequently also the scalars and vector fields (see Eq.[2.5] below). These multiplets have been obtained from the matrices $Q_\pm$, with the vacua in the spinor representations of the Lorentz algebra [4]. For instance, we have $\Xi_{++} = \begin{pmatrix} \Psi_{1+} \\ \bar{\Psi}_{2-} \\ \Psi_{3+} \end{pmatrix} \otimes \Omega_+$ with Ψ_{1+}, Ψ_{3+} two left-handed spinors, $\bar{\Psi}_{2-}$ a right handed spinor and Ω_+, the vacuum, a left-handed spinor. The transformation law for Ξ_{++} is then obtained from

$$
\delta_\varepsilon \Xi_{++} = \left(\varepsilon^m Q_{+m} \begin{pmatrix} \Psi_{1+} \\ \bar{\Psi}_{2-} \\ \Psi_{3+} \end{pmatrix} \right) \otimes \Omega_+ \tag{2.4}
$$

Similar definitions hold for the three other multiplets. We recall here the corresponding transformation laws obtained after some algebraic manipulation [4]

$$
\begin{array}{cc}
\overset{(+,+)}{\left\{\begin{array}{l}
\delta_\varepsilon \varphi = \varepsilon^m \tilde{A}_m \\
\delta_\varepsilon B_{mn} = -(\varepsilon_m \tilde{A}_n - \varepsilon_n \tilde{A}_m) + i\varepsilon_{mnpq}\varepsilon^p \tilde{A}^q \\
\delta_\varepsilon \tilde{A}_m = (\varepsilon_m \tilde{\tilde{\varphi}} + \varepsilon^n \tilde{\tilde{B}}_{mn}) \\
\delta_\varepsilon \tilde{\tilde{\varphi}} = \varepsilon^m \partial_m \varphi \quad \delta_\varepsilon \tilde{\tilde{B}}_{mn} = \varepsilon^p \partial_p B_{mn}
\end{array}\right.}
&
\overset{(+,-)}{\left\{\begin{array}{l}
\delta_\varepsilon A'_m = (\varepsilon^n \tilde{B}'_{mn} + \varepsilon_m \tilde{\varphi}') \\
\delta_\varepsilon \tilde{\varphi}' = \varepsilon^m \tilde{\tilde{A}}'_m \\
\delta_\varepsilon \tilde{B}'_{mn} = -(\varepsilon_m \tilde{\tilde{A}}'_n - \varepsilon_n \tilde{\tilde{A}}'_m) - i\varepsilon_{mnpq}\varepsilon^p \tilde{\tilde{A}}'^q \\
\delta \tilde{\tilde{A}}'_m = \varepsilon^n \partial_n A'_m
\end{array}\right.}
\\[2em]
\overset{(-,-)}{\left\{\begin{array}{l}
\delta_\varepsilon \varphi' = \varepsilon^m \tilde{A}'_m \\
\delta_\varepsilon B'_{mn} = -(\varepsilon_m \tilde{A}'_n - \varepsilon_n \tilde{A}'_m) - i\varepsilon_{mnpq}\varepsilon^p \tilde{A}'^q \\
\delta_\varepsilon \tilde{A}'_m = (\varepsilon_m \tilde{\tilde{\varphi}}' + \varepsilon^n \tilde{\tilde{B}}'_{mn}) \\
\delta_\varepsilon \tilde{\tilde{\varphi}}' = \varepsilon^m \partial_m \varphi' \ , \quad \delta_\varepsilon \tilde{\tilde{B}}'_{mn} = \varepsilon^p \partial_p B'_{mn}
\end{array}\right.}
&
\overset{(-,+)}{\left\{\begin{array}{l}
\delta_\varepsilon A_m = (\varepsilon^n \tilde{B}_{mn} + \varepsilon_m \tilde{\varphi}) \\
\delta_\varepsilon \tilde{\varphi} = \varepsilon^m \tilde{\tilde{A}}_m \\
\delta_\varepsilon \tilde{B}_{mn} = -(\varepsilon_m \tilde{\tilde{A}}_n - \varepsilon_n \tilde{\tilde{A}}_m) + i\varepsilon_{mnpq}\varepsilon^p \tilde{\tilde{A}}^q \\
\delta \tilde{\tilde{A}}_m = \varepsilon^n \partial_n A_m
\end{array}\right.}
\end{array}
\tag{2.5}
$$

with ε a real Lorentz vector and $P_m = \partial_m$ (this is a slight difference compared to [4], where ε was taken purely imaginary). As can be seen from the transformation laws (2.5), the complex conjugate of Ξ_{--} (resp. Ξ_{-+}) transforms like Ξ_{++} (resp. Ξ_{+-}). In the following we will thus consider the minimal set of field content, taking $\Xi^\star_{++} = \Xi_{--}, \Xi^\star_{+-} = \Xi_{-+}$ (*i.e.* $\varphi^\star = \varphi', \tilde{A}^\star = \tilde{A}'$ *etc.*), so that the multiplet Ξ_{ab} is the CPT conjugate of Ξ_{-a-b}.

We introduce for each $1-$form potential A_m the $2-$form field strength $F_{mn} = \partial_m A_n - \partial_n A_m$, and for each $2-$form potential a $3-$form field strength $H_{mnp} = \partial_m B_{np} + \partial_n B_{pm} + \partial_p B_{mn}$ together with its dual $1-$form ${}^\star H_m = \frac{1}{6}\varepsilon_{mnpq} H^{npq}$. We can construct two zero-graded real 3SUSY invariant Lagrangians associated respectively to the multiplets (Ξ_{++}, Ξ_{--}) and (Ξ_{+-}, Ξ_{-+}) [4]

$$
\begin{array}{rcl}
\mathcal{L}_0 &=& \mathcal{L}_0(\Xi_{++}) + \mathcal{L}_0(\Xi_{--}) \\
&=& \partial_m \varphi \partial^m \tilde{\tilde{\varphi}} + \dfrac{1}{12} H_{mnp} \tilde{\tilde{H}}^{mnp} - \dfrac{1}{2} {}^\star H_m {}^\star \tilde{\tilde{H}}^m - \dfrac{1}{4} \tilde{F}_{mn} \tilde{F}^{mn} - \dfrac{1}{2} \left(\partial_m \tilde{A}^m\right)^2 \\
&+& \partial_m \varphi' \partial^m \tilde{\tilde{\varphi}}' + \dfrac{1}{12} H'_{mnp} \tilde{\tilde{H}}'^{mnp} - \dfrac{1}{2} {}^\star H'_m {}^\star \tilde{\tilde{H}}'^m - \dfrac{1}{4} \tilde{F}'_{mn} \tilde{F}'^{mn} - \dfrac{1}{2} \left(\partial_m \tilde{A}'^m\right)^2 \\
\mathcal{L}'_0 &=& \mathcal{L}_0(\Xi_{-+}) + \mathcal{L}_0(\Xi_{+-}) \\
&=& \dfrac{1}{2} \partial_m \tilde{\varphi} \partial^m \tilde{\varphi} + \dfrac{1}{24} \tilde{H}_{mnp} \tilde{H}^{mnp} - \dfrac{1}{4} {}^\star \tilde{H}_m {}^\star \tilde{H}^m - \dfrac{1}{2} F_{mn} \tilde{\tilde{F}}^{mn} - (\partial_m A^m)(\partial_n \tilde{\tilde{A}}^n) \\
&+& \dfrac{1}{2} \partial_m \tilde{\varphi}' \partial^m \tilde{\varphi}' + \dfrac{1}{24} \tilde{H}'_{mnp} \tilde{H}'^{mnp} - \dfrac{1}{4} {}^\star \tilde{H}'_m {}^\star \tilde{H}'^m - \dfrac{1}{2} F'_{mn} \tilde{\tilde{F}}'^{mn} - (\partial_m A'^m)(\partial_n \tilde{\tilde{A}}'^n)
\end{array}
\tag{2.6}
$$

To identify the physical degrees of freedom, we proceed in several steps. We concentrate just on $\mathcal{L}_0$ *i.e* on the multiplets Ξ_{++} and Ξ_{--}. For $\mathcal{L}'_0$ the results are analogous.

Firstly, we introduce the real fields

$$
\begin{aligned}
\tilde{A}_1 &= \frac{\tilde{A}+\tilde{A}'}{\sqrt{2}} \quad , \quad \tilde{A}_2 = i\frac{\tilde{A}-\tilde{A}'}{\sqrt{2}}, \\
B_1 &= \frac{B+B'}{\sqrt{2}} \quad , \quad B_2 = i\frac{B-B'}{\sqrt{2}}, \\
\tilde{\tilde{B}}_1 &= \frac{\tilde{\tilde{B}}+\tilde{\tilde{B}}'}{\sqrt{2}} \quad , \quad \tilde{\tilde{B}}_2 = i\frac{\tilde{\tilde{B}}-\tilde{\tilde{B}}'}{\sqrt{2}}, \\
\varphi_1 &= \frac{\varphi+\varphi'}{\sqrt{2}} \quad , \quad \varphi_2 = i\frac{\varphi-\varphi'}{\sqrt{2}}, \\
\tilde{\tilde{\varphi}}_1 &= \frac{\tilde{\tilde{\varphi}}+\tilde{\tilde{\varphi}}'}{\sqrt{2}} \quad , \quad \tilde{\tilde{\varphi}}_2 = i\frac{\tilde{\tilde{\varphi}}-\tilde{\tilde{\varphi}}'}{\sqrt{2}}.
\end{aligned}
\tag{2.7}
$$

Then, using

$$
{}^\star B_1 = B_2, {}^\star\tilde{\tilde{B}}_1 = \tilde{\tilde{B}}_2, \tag{2.8}
$$

the Lagrangian $\mathcal{L}_0$ becomes

$$
\begin{aligned}
\mathcal{L}_0 &= \partial_m\varphi_1\partial^m\tilde{\tilde{\varphi}}_1 - \partial_m\varphi_2\partial^m\tilde{\tilde{\varphi}}_2 + \frac{1}{6}H_{1mnp}\tilde{\tilde{H}}_1^{\ mnp} + \partial^n B_{1nm}\partial_p\tilde{\tilde{B}}_1{}^{pm} \\
&- \frac{1}{4}\tilde{F}_{1mn}\tilde{F}_1{}^{mn} + \frac{1}{4}\tilde{F}_{2mn}\tilde{F}_2{}^{mn} - \frac{1}{2}\left(\partial_m\tilde{A}_1{}^m\right)^2 + \frac{1}{2}\left(\partial_m\tilde{A}_2{}^m\right)^2.
\end{aligned}
\tag{2.9}
$$

We observe that we have started with two self-dual $2-$forms $B, \tilde{\tilde{B}}$ and two anti-self-dual $2-$forms $B', \tilde{\tilde{B}}'$ and we end up with two $2-$forms $B_1, \tilde{\tilde{B}}_1$. The $2-$forms $B_2, \tilde{\tilde{B}}_2$ are related to the $2-$forms $B_1, \tilde{\tilde{B}}_1$ by duality transformations (2.8) and thus they do not appear in the Lagrangian. These final $2-$forms are neither self-dual nor anti-self-dual, which is in perfect agreement with the theory of representations of the $4D-$Poincaré group [1].

Secondly, we observe that the terms in the first line of (2.9) are not diagonal. Thus, we define

$$
\begin{aligned}
\hat{\varphi}_1 = \frac{\varphi_1+\tilde{\tilde{\varphi}}_1}{\sqrt{2}}, \hat{\hat{\varphi}}_1 &= \frac{\varphi_1-\tilde{\tilde{\varphi}}_1}{\sqrt{2}}, \hat{\varphi}_2 = \frac{\varphi_2+\tilde{\tilde{\varphi}}_2}{\sqrt{2}}, \hat{\hat{\varphi}}_2 = \frac{\varphi_2-\tilde{\tilde{\varphi}}_2}{\sqrt{2}}, \\
\hat{B}_1 &= \frac{B_1+\tilde{\tilde{B}}_1}{\sqrt{2}}, \hat{\hat{B}}_1 = \frac{B_1-\tilde{\tilde{B}}_1}{\sqrt{2}},
\end{aligned}
\tag{2.10}
$$

and $\mathcal{L}_0$ reduces to

[1] In the representation theory of $SO(1,3)$, the $2-$forms are either self- or anti-self-dual. For the Poincaré group, in the massless case where the little group is $SO(2)$, it is the $1-$forms that are self or anti-self-dual (in the case of the electromagnetism these two possibilities correspond to the two polarisations of the photon)

$$
\begin{aligned}
\mathcal{L}_0 &= \frac{1}{2}\partial_m\hat{\varphi}_1\partial^m\hat{\varphi}_1 - \frac{1}{2}\partial_m\hat{\hat{\varphi}}_1\partial^m\hat{\hat{\varphi}}_1 - \frac{1}{2}\partial_m\hat{\varphi}_2\partial^m\hat{\varphi}_2 + \frac{1}{2}\partial_m\hat{\hat{\varphi}}_2\partial^m\hat{\hat{\varphi}}_2 \\
&- \frac{1}{2}\partial_m\tilde{A}_{1n}\partial^m\tilde{A}_1{}^n + \frac{1}{2}\partial_m\tilde{A}_{2n}\partial^m\tilde{A}_2{}^n \\
&+ \frac{1}{4}\partial_m\hat{B}_{1np}\partial^m\hat{B}_1{}^{np} - \frac{1}{4}\partial_m\hat{\hat{B}}_{1np}\partial^m\hat{\hat{B}}_1{}^{np}.
\end{aligned}
\tag{2.11}
$$

which we now express only in terms of the potentials, including the contribution of the gauge fixings terms of (2.9). We observe that the kinetic terms for $\hat{\hat{\varphi}}_1, \hat{\varphi}_2, \tilde{A}_2, \hat{\hat{B}}_1$ have wrong relative signs. We will come back to this point in the next subsection.

As it has been noted previously in [4], P^2 is a Casimir operator, and thus all states in an irreducible representation have the same mass m. An invariant mass term for each multiplets in (2.3) can thus be added to the Lagrangian. For instance

$$
\mathcal{L}[\Xi_{++}]_{\text{mass}} = m^2(\varphi\tilde{\tilde{\varphi}} + \frac{1}{4}B^{mn}\tilde{\tilde{B}}_{mn} - \frac{1}{2}\tilde{A}_m\tilde{A}^m), \tag{2.12}
$$

where the mass m could be related to the parameter Λ appearing in (2.2). Finally, let us also note that a term like

$$
\mathcal{L}_\varphi = g\tilde{\tilde{\varphi}} \tag{2.13}
$$

is invariant on its own. We see that this term is of gradation 1 which is not the case for (2.6), (2.12) and all the other Lagrangians considered here.

A last comment regarding the 2-forms is in order. In (2.6), the 2-forms are self-dual or anti-self-dual, so that a usual gauge transformation which does not preserve their (anti)-self-dual character cannot be applied. The status of gauge fixing through terms of the type $\frac{1}{12}H_{mnp}\tilde{\tilde{H}}^{mnp} - \frac{1}{2}{}^\star H_m{}^\star\tilde{\tilde{H}}^m$ in (2.6) is therefore not explicit. After performing the change(s) of variables (2.7, 2.10), the 2-forms are now neither self-dual nor anti-self-dual, the usual gauge transformations become well defined and the gauge fixing for the 2-forms in (2.9) (or (2.11)) is transparent. We will come back to this point in more details in section 4.

2.2 Dualisation

In this subsection, we propose a possible way to construct a Lagrangian with correct signs for the various kinetic terms, based on a special choice for the physical fields. The main idea is related to Hodge duality. However, the duality transformation will act here on the $p-$forms with respect to the Lorentz group $SO(1,3)$. This should be contrasted with the case of the usual duality transformations (generalising the electric-magnetic duality) which act on the field strengths with respect to $SO(1,3)$, or equivalently on the potentials themselves but with respect to the little group $SO(2)$.

To simplify, we use the notations of differential forms. Introducing the exterior derivative d which maps a $p-$form into a $(p+1)-$form, and its adjoint $d^\dagger$ which maps a $p-$form into a $(p-1)-$form, we have
for a $0-$form, say $\hat{\hat{\varphi}}_1$

$$\frac{1}{2}\partial_m\hat{\hat{\varphi}}_1\partial^m\hat{\hat{\varphi}}_1 = \frac{1}{2}d\hat{\hat{\varphi}}_1 d\hat{\hat{\varphi}}_1$$

for a $1-$form, say A_2

$$\frac{1}{4}F_{2mn}F_2{}^{mn} + \frac{1}{2}\left(\partial_m A_2{}^m\right)^2 = \frac{1}{4}dA_2 dA_2 + \frac{1}{2}d^\dagger A_2 d^\dagger A_2$$

for a $2-$form, say $\hat{B}_1$

$$\frac{1}{12}\hat{H}_{1mnp}\hat{H}_1{}^{mnp} + \frac{1}{2}\partial^n\hat{B}_{1nm}\partial_p\hat{B}_1{}^{pm} = \frac{1}{12}d\hat{B}_1 d\hat{B}_1 + \frac{1}{2}d^\dagger\hat{B}_1 d^\dagger\hat{B}_1.$$

Using, for a given $p-$form $A_{[p]}$,

$$\begin{aligned} & \frac{1}{(p+1)!}dA_{[p]}dA_{[p]} + \frac{1}{(p-1)!}d^\dagger A_{[p]}d^\dagger A_{[p]} \\ = & -\left(\frac{1}{(4-p-1)!}d^\dagger B_{[4-p]}d^\dagger B_{[4-p]} + \frac{1}{(4-p+1)!}dB_{[4-p]}dB_{[4-p]}\right) \end{aligned} \tag{2.14}$$

with $B_{[4-p]} = {}^\star A_{[p]}$, and introducing

$$\begin{array}{ll} \hat{\hat{D}}_1 = {}^\star\hat{\hat{\varphi}}_1, \hat{D}_2 = {}^\star\hat{\varphi}_2, & 4-\text{forms}, \\ \tilde{C}_2 = {}^\star\tilde{A}_2, & 3-\text{form}, \\ \hat{\hat{\mathcal{B}}}_1 = {}^\star\hat{\hat{B}}_1, & 2-\text{form}, \end{array} \tag{2.15}$$

we observe that their corresponding kinetic terms have the correct sign. This means that the physical fields are not $\hat{\hat{\varphi}}_1, \hat{\varphi}_2, \tilde{A}_2, \hat{\hat{B}}_1$ but their Hodge duals $\hat{\hat{D}}_1, \hat{D}_2, \tilde{C}_2, \hat{\hat{\mathcal{B}}}_1$. The transformation (2.14) is possible due to the specific form of our Lagrangian which contains usual kinetic terms plus gauge fixing terms. In our transformations, $A_{[p]} \to B_{[4-p]}$, the kinetic term of A becomes the gauge fixing term of B and *vice-versa*. This is our duality symmetry. In the case of the $0-$form (resp. $4-$form), we only have a kinetic (resp. gauge fixing) term.

At the very end the Lagrangian writes

$$\begin{aligned} \tilde{\mathcal{L}}_0 = & \frac{1}{2}d\hat{\varphi}_1 d\hat{\varphi}_1 + \frac{1}{2}d\hat{\hat{\varphi}}_2 d\hat{\hat{\varphi}}_2 \\ - & \frac{1}{4}d\tilde{A}_1 d\tilde{A}_1 - \frac{1}{2}d^\dagger\tilde{A}_1 d^\dagger\tilde{A}_1 \\ + & \frac{1}{12}d\hat{B}_1 d\hat{B}_1 + \frac{1}{2}d^\dagger\hat{B}_1 d^\dagger\hat{B}_1 + \frac{1}{12}d\hat{\hat{\mathcal{B}}}_1 d\hat{\hat{\mathcal{B}}}_1 + \frac{1}{2}d^\dagger\hat{\hat{\mathcal{B}}}_1 d^\dagger\hat{\hat{\mathcal{B}}}_1 \\ - & \frac{1}{48}d\tilde{C}_2 d\tilde{C}_2 - \frac{1}{4}d^\dagger\tilde{C}_2 d^\dagger\tilde{C}_2 \\ + & \frac{1}{12}d^\dagger\hat{\hat{D}}_1 d^\dagger\hat{\hat{D}}_1 + \frac{1}{12}d^\dagger\hat{D}_2 d^\dagger\hat{D}_2 \end{aligned} \tag{2.16}$$

with the physical degrees of freedom as follows: in the sector of gradation -1 and 1 two $0-$forms $(\hat{\varphi}_1, \hat{\hat{\varphi}}_2)$, two $2-$forms $(\hat{B}_1, \hat{\hat{\mathcal{B}}}_1)$ and two $4-$forms $(\hat{\hat{D}}_1, \hat{D}_2)$; in the zero-graded sector one $1-$form $\tilde{A}_1$ and one $3-$form $\tilde{C}_2$. We note that the physical states are mixtures of states belonging to two CPT-conjugate multiplets (2.7) and also mixtures of the graded $(-1)-$ and the graded $1-$sectors (2.10).

Considering the gauge transformation for a $p-$form $A_{[p]}$, $(p \geq 1)$,

$$A_{[p]} \rightarrow A_{[p]} + d\chi_{[p-1]}, \tag{2.17}$$

where $\chi_{[p-1]}$ a $(p-1)-$form, the presence of terms involving $d^\dagger$ in the Lagrangian fixes partially the gauge to $d^\dagger d\chi_{[p-1]} = 0$. This means that the terms involving the $d^\dagger$ operators can be seen as some Feynman gauge fixing terms adapted for $p-$forms. Another way of seeing this phenomenon is to rewrite $\mathcal{L}_0$ with Fermi-like terms $(\frac{1}{p!}\partial A_{[p]}\partial A_{[p]})$. It is well known that $A_{[p]}, 0 \leq p \leq 2$ give rise to a massless state in the $p-$order antisymmetric representation of the little group $SO(2)$. But, in our decomposition, there are also $p-$forms with $p = 3, 4$. [It is interesting to note that similar phenomena are well-known in the context of type IIA, IIB string theory [5] in 10 space-time dimensions where $9-$ and $10-$forms appear. Actually, subsequent to the early works on two-forms in [6, 7], several authors studied the classical and quantum properties of the non-propagating 3- and 4-forms [8, 9]. In particular, it was pointed out in [9] that the gauge fixing term for a 4-form takes the form of a kinetic term for a scalar field, in exact analogy with our results.]

Before ending this section let us make some comments on the number of degrees of freedom of the various fields and the role played by the gauge fixing terms [more details are given in section 4]. We should first stress the difference between our case and the conventional gauge invariant theories, despite the presence of the "familiar" gauge fixing terms of the form $(d^\dagger A_{[p]})^2$ in (2.16). Indeed, while in the case of gauge theories the gauge invariance guarantees that the physical (on-shell) quantities are gauge fixing independent, in our case $(d^\dagger A_{[p]})^2$ cannot be traded for any other gauge-fixing function since it is imposed by the 3SUSY invariance, and is thus expected to affect the physical degrees of freedom. Let us illustrate the point on a generic Lagrangian of the form $L_A = (dA_{[p]})^2 + (d^\dagger A_{[p]})^2$ which, apart from relative coefficients which are unimportant for the discussion, is the one dictated by the 3SUSY invariance for $p = 1, 2, 3$. L_A has a restricted invariance under $A_{[p]} \rightarrow A_{[p]} + d\chi_{[p-1]}$ and $A_{[p]} \rightarrow A_{[p]} + d^\dagger\chi_{[p+1]}$ for the subclasses of forms satisfying $d^\dagger d\chi_{[p-1]} = 0$ and $dd^\dagger\chi_{[p+1]} = 0$. However due to Poincaré's theorem (barring topological effects which we do not consider in this paper) the latter constraint on $\chi_{[p+1]}$ implies that there exists a $(p-1)-$form $\lambda_{[p-1]}$ such that $d^\dagger\chi_{[p+1]} = d\lambda_{[p-1]}$. Hence the second invariance of L_A is actually also of the gauge type with the constraint $d^\dagger d\lambda_{[p-1]} = 0$ and does not correspond to an extra freedom.[2] This shows that the effective degrees of freedom of $A_{[p]}$ are dictated only by the gauge freedom eq.(2.17), supplemented by $d^\dagger d\chi_{[p-1]} = 0$. An immediate consequence of the latter constraint is that the usual Lorentz condition $d^\dagger A_{[p]} = 0$ cannot be imposed in general to eliminate

[2]Note that an equivalent formulation holds if one considers the constraint on $\chi_{[p-1]}$, and amounts to interchanging the roles of d and $d^\dagger$.

the unphysical components. This means that the way one should eliminate the unphysical components cannot be handled in a usual manner [10]. On top of that, such a condition is not stable under our transformation laws (2.5). [For instance, if we put $\partial_m \tilde{A}^m = 0$, then $\partial_m \delta_\varepsilon \tilde{A}^m = 0$ gives $\varepsilon^m \partial_m \tilde{\tilde{\varphi}} = 0$, which is obviously too strong.] Furthermore, it should be clear that the constraint $d^\dagger d\chi_{[p-1]} = 0$ on the gauge generators $\chi_{[p-1]}$ is not be confused with the dependences among the gauge transformations on $A_{[p]}$ which originate when these transformations possess themselves some gauge invariance (see for instance [11] for a review). Off-shell, the usual gauge invariance would have lead to 3 degrees of freedom for each $1-$ and $2-forms$, and 1 degree of freedom for the $3-$form. Recall that this conventional counting corresponds to the maximal elimination of gauge redundancies *assuming that the field components have arbitrarily general forms*. (for instance, particular configurations such as fields of the pure gauge form can be completely gauged away.) This point is of particular relevance to our case: the equation $d^\dagger d\chi_{[p-1]} = 0$ reduces the space of allowed space-time configurations of $\chi_{[p-1]}$. The elimination of redundant degrees of freedom in $A_{[p]}$ are thus possible only when the space-time configurations of the latter are consistent with those of $\chi_{[p-1]}$. Thus if we insist on having arbitrary configurations for the components of $A_{[p]}$, then the residual gauge invariance does not eliminate any degree of freedom, *i.e.* $\tilde{A}$, $\tilde{C}$ have each 4 degrees of freedom, $\hat{\hat{\mathcal{B}}}$ and $\hat{B}$ 6 degrees of freedom each and φ's and D's one degree of freedom. However the situation is not as simple, since on the one hand the 3SUSY could itself impose some space-time configuration constraints on the components of a given 3SUSY multiplet, and on the other hand the gauge transformation should preserve the 3SUSY character of the transformed fields. We will come back to these issues in more detail in section 4.

2.3 Mixing between different multiplets

We can now consider coupling terms between different multiplets. The basic idea is to couple different types of multiplets such that zero-graded couplings between a potential and a field strength are possible. Having this in mind, one can *a priori* couple the multiplet Ξ_{++} with either Ξ_{+-} or Ξ_{-+} (2.3). Imposing the 3SUSY invariance, Ξ_{++} can only be coupled with Ξ_{+-} (or Ξ_{--} with Ξ_{-+}). We name these two pairs of multiplets *interlaced multiplets*.

For the sake of simplicity, we do not consider hereafter the fields introduced in the previous subsection, keeping in mind that the dualisation (2.15) can be performed at any step if necessary.

The simplest Lagrangian mixing Ξ_{++} with Ξ_{+-} and Ξ_{--} with Ξ_{-+}, expressed with the fields appearing in (2.3), is

$$\begin{aligned}
\mathcal{L}_{\rm c} &= \mathcal{L}_{\rm c}(\Xi_{++}, \Xi_{+-}) + \mathcal{L}_{\rm c}(\Xi_{--}, \Xi_{-+}) \\
&= \lambda \left(\partial_m \varphi \tilde{\tilde{A}}'^m + \partial_m \tilde{\tilde{\varphi}} A'^m - \partial_m \tilde{A}^m \tilde{\varphi}' - \partial_m \tilde{A}_n \tilde{B}'^{mn} + \partial^m B_{mn} \tilde{\tilde{A}}'^n + \partial^m \tilde{\tilde{B}}_{mn} A'^n \right) \\
&+ \lambda^\star \left(\partial_m \varphi' \tilde{\tilde{A}}^m + \partial_m \tilde{\tilde{\varphi}}' A^m - \partial_m \tilde{A}'^m \tilde{\varphi} - \partial_m \tilde{A}'_n \tilde{B}^{mn} + \partial^m B'_{mn} \tilde{\tilde{A}}^n + \partial^m \tilde{\tilde{B}}'_{mn} A^n \right)
\end{aligned} \tag{2.18}$$

with $\lambda = \lambda_1 + i\lambda_2$ a complex coupling constant with mass dimension. Due to the CPT con-

jugation, the Lagrangian $\mathcal{L}_c$ is real. We emphasize here that if terms of non-zero gradation were included, they would have had to be separately invariant as can be seen from (2.5). Furthermore, one can explicitly check that there is no invariant Lagrangian which is bilinear in the fields and of gradation 1 or (-1). This is in perfect agreement with the results of the next section.

To show the invariance of $\mathcal{L}_c$ it is sufficient to study separately the two parts $\mathcal{L}_c(\Xi_{++}, \Xi_{+-})$ and $\mathcal{L}_c(\Xi_{--}, \Xi_{-+})$, since they do not mix under 3SUSY transformations. From (2.5) one finds

$$\delta_\varepsilon \mathcal{L}_c(\Xi_{++}, \Xi_{+-}) \hat{=} -\frac{1}{4}\lambda \tilde{B}'^{mn}\left(\varepsilon^r \partial_{[m}\tilde{\tilde{B}}_{n]_- r} - \varepsilon_{[m}\partial^r \tilde{\tilde{B}}_{n]_- r}\right) \tag{2.19}$$

where we used the shorthand notation $X_{[mn]_-}$ for the anti-self-dualised 2−form (see eq.(3.5)), (the hatted equality denotes equality up to surface terms). After some algebraic manipulations (see section **3.1**) one finds

$$\varepsilon^r \partial_{[m}\tilde{\tilde{B}}_{n]_- r} - \varepsilon_{[m}\partial^r \tilde{\tilde{B}}_{n]_- r} = 0,$$

so that $\delta_\varepsilon \mathcal{L}_c(\Xi_{++}, \Xi_{+-})$ reduces to a total derivative. A similar result holds for $\mathcal{L}_c(\Xi_{--}, \Xi_{-+})$, thus (2.18) is an invariant Lagrangian.

Let us now consider the total Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}'_0 + \mathcal{L}_c, \tag{2.20}$$

where $\mathcal{L}_0$ and $\mathcal{L}'_0$ are given in (2.6) and $\mathcal{L}_c$ given in (2.18). Since $\mathcal{L}$ is quadratic in the fields, it is always possible, by a field redefinition, to rewrite $\mathcal{L}$ in a diagonal form.

To proceed with this calculation, we perform the change of variables defined in (2.7) and (2.10) to cast $\mathcal{L}_0$ in a diagonal form, (and of course similar transformations to $\mathcal{L}'_0$). Direct inspection shows that $\mathcal{L}$ contains 15 fields:

6 scalar fields, $\hat{\varphi}_1, \hat{\hat{\varphi}}_1, \hat{\varphi}_2, \hat{\hat{\varphi}}_2$ (in $\mathcal{L}_0$), $\tilde{\varphi}_1, \tilde{\tilde{\varphi}}_2$ (in $\mathcal{L}'_0$);

6 vector fields, $\tilde{A}_1, \tilde{\tilde{A}}_2$ (in $\mathcal{L}_0$), $\hat{A}_1, \hat{\hat{A}}_1, \hat{A}_2, \hat{\hat{A}}_2$ (in $\mathcal{L}'_0$);

3 two-forms $\hat{B}_1, \hat{\hat{B}}_1$ (in $\mathcal{L}_0$) $\tilde{B}_2$ (in $\mathcal{L}'_0$).

The notations for the fields of $\mathcal{L}'_0$ follow the same logic as the notations of $\mathcal{L}_0$. In order to diagonalise $\mathcal{L}$ one observes that we have three decoupled Lagrangians:

$$\mathcal{L} = \mathcal{L}_1(\hat{\varphi}_1, \hat{\varphi}_2, \hat{A}_1, \hat{A}_2, \hat{B}_1) + \mathcal{L}_2(\hat{\hat{\varphi}}_1, \hat{\hat{\varphi}}_2, \hat{\hat{A}}_1, \hat{\hat{A}}_2, \hat{\hat{B}}_1) + \mathcal{L}_3(\tilde{\varphi}_1, \tilde{\tilde{\varphi}}_2, \tilde{A}_1, \tilde{\tilde{A}}_2, \tilde{B}_1). \tag{2.21}$$

with $\mathcal{L}_1$,$\mathcal{L}_2$ and $\mathcal{L}_3$ having the same form. We will explicitly consider one of them that we denote generically $\mathcal{L}(\varphi_1, \varphi_2, A_1, A_2, B)$:

$$\begin{aligned}
\mathcal{L}(\varphi_1, \varphi_2, A_1, A_2, B) \;&=\; \frac{1}{2}(\partial_m\varphi_1)^2 - \frac{1}{2}(\partial_m\varphi_2)^2 - \frac{1}{2}(\partial_m A_{1n})^2 + \frac{1}{2}(\partial_m A_{2n})^2 + \frac{1}{4}(\partial_m B_{np})^2 \\
&+\; \lambda_1\left({A_1}^m\partial_m\varphi_1 + {A_2}^m\partial_m\varphi_2 - B^{mn}\partial_m A_{1n} - {}^\star B^{mn}\partial_m A_{2n}\right) \qquad (2.22) \\
&+\; \lambda_2\left(-{A_2}^m\partial_m\varphi_1 + {A_1}^m\partial_m\varphi_2 + B^{mn}\partial_m A_{2n} - {}^\star B^{mn}\partial_m A_{1n}\right).
\end{aligned}$$

Let us comment at this level some of the terms appearing in this Lagrangian. The last two lines of (2.22), which originate from from (2.18), contain exactly the same gauge fixing as in (2.11). For B, only $\frac{1}{2}B^{mn}F_{imn}$ ($i = 1, 2$) fix the gauge, while ${}^\star B^{mn}F_{imn}$, $i = 1, 2$ are gauge invariant. Terms like ${}^\star B^{mn}F_{mn}$ (called $BF-$terms) related to topological theories where initially introduced in [12]. Their natural appearance within 3SUSY may suggest some underlying topological properties whose study is, however, out of the track of the present paper. Other mixing terms between the A and φ fields are of the Goldstone type which appear after spontaneous symmetry breaking, however, in our case such terms cannot be gauged away since the gauge is already partially fixed.

In order to diagonalise (2.22) we proceed in several steps. (Of course in addition to (2.22) one can add the mass terms (2.12). But for simplicity we do not consider them here.) Firstly, we express the action in Fourier space. Secondly, we construct some perfect squares for the terms involving A_1 and then A_2. After a tedious calculation, we obtain

$$\begin{aligned}
\tilde{\mathcal{L}} &= \frac{1}{2}\left(p^2-(\lambda_2^2-\lambda_1^2)\right)\tilde{\varphi}_1(p)\tilde{\varphi}_1(-p) - \frac{1}{2}\left(p^2-(\lambda_2^2-\lambda_1^2)\right)\tilde{\varphi}_2(p)\tilde{\varphi}_2(-p) \\
&+ \lambda_1\lambda_2\left(\tilde{\varphi}_1(p)\tilde{\varphi}_2(-p)+\tilde{\varphi}_2(p)\tilde{\varphi}_1(-p)\right) \\
&- \frac{1}{2}p^2\tilde{A}'_{1m}(p)\tilde{A}'^{\,m}_1(-p) + \frac{1}{2}p^2\tilde{A}'_{2m}(p)\tilde{A}'^{\,m}_2(-p) + \frac{1}{4}p^2\tilde{B}_{mn}(p)\tilde{B}^{mn}m(-p) \qquad (2.23) \\
&+ \frac{1}{2}\frac{1}{p^2}p^r p_s(\lambda_1^2-\lambda_2^2)\left(\tilde{B}_{rm}(p)\tilde{B}^{sm}(-p) - {}^\star\tilde{B}_{rm}(p){}^\star\tilde{B}^{sm}(-p)\right) \\
&+ \frac{\lambda_1\lambda_2}{p^2}p^r p_s\left(\tilde{B}_{rm}(p){}^\star\tilde{B}^{sm}(-p) + \tilde{B}^{sm}(p){}^\star\tilde{B}_{rm}(-p)\right),
\end{aligned}$$

where the tilde denotes the Fourier transform (not to be confused with the tilde in the fields we had until (2.21)) and

$$\begin{aligned}
\tilde{A}'_{1m}(p) &= \tilde{A}_{1m}(p) + \frac{\lambda_1}{p^2}ip_m\tilde{\varphi}_1(p) + \frac{\lambda_2}{p^2}ip_m\tilde{\varphi}_2(p) + \frac{\lambda_1}{p^2}ip^r\tilde{B}_{rm}(p) + \frac{\lambda_2}{p^2}ip^r({}^\star\tilde{B}_{rm}(p)) \qquad (2.24) \\
\tilde{A}'_{2m}(p) &= \tilde{A}_{2m}(p) - \frac{\lambda_1}{p^2}ip_m\tilde{\varphi}_2(p) + \frac{\lambda_2}{p^2}ip_m\tilde{\varphi}_1(p) + \frac{\lambda_2}{p^2}ip^r\tilde{B}_{rm}(p) - \frac{\lambda_1}{p^2}ip^r({}^\star\tilde{B}_{rm}(p)).
\end{aligned}$$

A simple transformation diagonalises the φ part of the Lagrangian. Then, in order not to have unwanted tachyons we need to impose $\lambda_2^2 \geq \lambda_1^2$. Finally, the kinetic part for the B field is non conventional. Nevertheless, taking $\lambda_1 = 0$ simplifies somewhat the Lagrangian. As expected, the wrong signs of some of the kinetic terms do not change by this diagonalisation. However, we could have proceeded along the same lines with the fields given in section 2.2 where the Lagrangian involves one $0-$, one $1-$, one $2-$, one $3-$ and one $4-$form fields.

3 Interactions

In the previous section only quadratic terms describing freely propagating fields (albeit with some non-trivial mixing) were considered. To construct interactions one must consider higher order terms. We will show in this section that such terms, describing interactions among

the four multiplets Ξ, are forbidden by 3SUSY. Such an obstruction is welcome, at least at the level of dimension four operators, in order to maintain the physical interpretation of the compatibility between 3SUSY and the $U(1)$ gauge symmetry in terms of gauge fixing of the latter. Indeed, such an interpretation would be lost if the 3SUSY allowed dimension four self-interactions between the gauge fields, which would then break further the $U(1)$ gauge symmetry.

3.1 Derivative multiplets

In this subsection we state some useful properties which will be repeatedly used in the rest of the paper

Self-dualities: denoting generic (anti)-self-dual 2-forms by $(R^{(-)})$ $R^{(+)}$, one has

$$\frac{1}{2}\varepsilon_{mnpq}R^{(\pm)pq} = \pm i R^{(\pm)}_{mn} \tag{3.1}$$

leading to the following relations

$$\begin{aligned}
R^{(\mp)}_{pq}R^{(\pm)pq} &= 0 \\
R^{(\pm)}_{mr}R^{(\pm)}{}_n{}^r &= \frac{1}{4}\eta_{mn}R^{(\pm)}_{pq}R^{(\pm)pq} \\
\varepsilon_{mnpr}R^{(\pm)}{}_q{}^r &= \pm i(\,\eta_{mq}R^{(\pm)}_{np} + \eta_{nq}R^{(\pm)}_{pm} + \eta_{pq}R^{(\pm)}_{mn}\,).
\end{aligned} \tag{3.2}$$

Furthermore, defining partial derivatives with respect to $R^{(\pm)}_{mr}x^r$,

$$\bar{\partial}^m_{(\pm)} \equiv \frac{\partial^m}{\partial R^{(\pm)}_{mr}x^r}$$

one has

$$\begin{aligned}
\bar{\partial}^m_{(\pm)} &= \frac{4}{R^{(\pm)}_{pq}\ R^{(\pm)pq}}R^{(\pm)m}{}_r\ \partial^r \\
\overline{\Box}_{(\pm)} &= \frac{4}{R^{(\pm)}_{pq}\ R^{(\pm)pq}}\ \Box
\end{aligned} \tag{3.3}$$

as a consequence of the second equation in (3.2) (provided that $R^{(\pm)}_{pq}\ R^{(\pm)pq} \neq 0$). Finally, the third equation in (3.2) leads immediately to

$$\begin{aligned}
\varepsilon_{mnpr}\,\partial^q\,R^{(\pm)}{}_q{}^r &= \pm i(\,\partial_m R^{(\pm)}_{np} + \partial_n R^{(\pm)}_{pm} + \partial_p R^{(\pm)}_{mn}\,) \\
\varepsilon_{mnpr}\,\partial^p\,R^{(\pm)}{}_q{}^r &= \pm i(\,\partial_q R^{(\pm)}_{mn} + \eta_{qm}\partial_r R^{(\pm)}{}_n{}^r - \eta_{qn}\partial_r R^{(\pm)}{}_m{}^r\,).
\end{aligned} \tag{3.4}$$

We also denote the (anti)-self-dualisation of any second rank tensor X_{mn} by

$$X_{[mn]_\pm} \equiv X_{mn} - X_{nm} \mp i\varepsilon_{mnpq}X^{pq}. \tag{3.5}$$

Derivative multiplets: For each multiplet of a given type Ξ, (2.3) we define its derivative $\mathcal{D}\,\Xi$ by saturating properly the Lorentz indices and combining components respecting the $\mathbb{Z}_3-$gradation as well as the self-duality properties in the following way:
for $\Xi_{\pm\mp} \equiv (A_m,\ \tilde{\varphi}, \tilde{B}_{mn},\ \tilde{\tilde{A}}_m)$ one constructs

$$\begin{aligned} \mathcal{D}\,\Xi_{\pm\mp} &= \Big(\psi, \psi_{mn},\ \tilde{\psi}_m,\ \tilde{\tilde{\psi}}, \tilde{\tilde{\psi}}_{mn}\Big) \\ &\equiv \Big(\partial_m A^m, \partial_{[m}A_{n]\pm},\ \partial_m\tilde{\varphi} + \partial^n \tilde{B}^{(\mp)}_{nm},\ \partial_m \tilde{\tilde{A}}^m, \partial_{[m}\tilde{\tilde{A}}_{n]\pm}\Big). \end{aligned} \tag{3.6}$$

Similarly, for $\Xi_{\pm\pm} \equiv (\varphi, B_{mn},\ \tilde{A}_m,\ \tilde{\tilde{\varphi}}, \tilde{\tilde{B}}_{mn})$,

$$\begin{aligned} \mathcal{D}\,\Xi_{\pm\pm} &= \Big(\psi_m,\ \tilde{\psi}, \tilde{\psi}_{mn},\ \tilde{\tilde{\psi}}_m\Big) \\ &\equiv \Big(\partial_m\varphi + \partial^n B^{(\pm)}_{nm},\ \partial_m\tilde{A}^m, \partial_{[m}\tilde{A}_{n]\mp},\ \partial_m\tilde{\tilde{\varphi}} + \partial^n \tilde{\tilde{B}}^{(\pm)}_{nm}\Big) \end{aligned} \tag{3.7}$$

The transformation laws for $\mathcal{D}\Xi$ are rather straightforward to establish using (2.5). For instance, one obtains for $\mathcal{D}\Xi_{\pm\mp}$

$$\begin{aligned} &\delta_\varepsilon\psi = \varepsilon^m\tilde{\psi}_m, \quad \delta_\varepsilon\tilde{\tilde{\psi}} = \varepsilon^m\partial_m\psi, \quad \delta_\varepsilon\tilde{\psi}^m = \varepsilon^m\tilde{\tilde{\psi}} + \varepsilon_n\tilde{\tilde{\psi}}^{mn} \\ &\delta_\varepsilon\psi_{mn} = \varepsilon_n\tilde{\psi}_m - \varepsilon_m\tilde{\psi}_n + i\varepsilon_{mnpq}e^p\tilde{\psi}^q + (\varepsilon^r\partial_{[n}\tilde{B}^{(\mp)}_{m]\pm r} - \varepsilon_{[n}\partial^r\tilde{B}^{(\mp)}_{m]\pm r}) \\ &\delta_\varepsilon\tilde{\tilde{\psi}}_{mn} = \varepsilon^r\partial_r\psi_{mn} \end{aligned} \tag{3.8}$$

showing that $\mathcal{D}\Xi_{(\pm\mp)}$ transforms like a $(\pm,\pm)$ multiplet provided that

$$\varepsilon^r\partial_{[n}\tilde{B}^{(\mp)}_{m]\pm r} - \varepsilon_{[n}\partial^r\tilde{B}^{(\mp)}_{m]\pm r} = 0. \tag{3.9}$$

This last equation is indeed satisfied as can be shown by using the two relations in (3.4). A similar result holds for the transformation laws of $\mathcal{D}\,\Xi_{\pm\pm}$. We thus have the following important property,

I: *The derivative of any multiplet of the type* $(s\mp)$ *is a multiplet of the type* $(s\pm)$.

$$\mathcal{D}\,\Xi_{s\mp} \sim \Xi'_{s\pm} \tag{3.10}$$

(where $s = +$ or $-$).

The two next subsections will be devoted to the proof that cubic supersymmetry forbids any interaction terms for the considered multiplets.

3.2 Tensor calculus

A natural way to build invariant interaction Lagrangians would be to define a tensor calculus which allows to construct a 3SUSY multiplet starting from two or more multiplets Ξ of any of the four types defined in (2.3)[3]. We will show here that if one starts from two arbitrary

[3]Recall that such techniques were initially used in the case of supersymmetry before the superspace formulation [13].

multiplets Ξ, it will not be possible to construct quadratically a third one of any of the four types defined in (2.3). In such a systematic study, one has to consider all possible triplets of multiplets. We exemplify this here on the specific case of multiplets of the type Ξ_{++}: starting from $\Xi_{1++} = \big(\varphi_1, B_1, \tilde{A}_1, \tilde{\tilde{\varphi}}_1, \tilde{\tilde{B}}_1\big)$ and $\Xi_{2++} = \big(\varphi_2, B_2, \tilde{A}_2, \tilde{\tilde{\varphi}}_2, \tilde{\tilde{B}}_2\big)$ we seek for a third one of the same type, $\Xi_{12++} = \big(\varphi_{12}, B_{12}, \tilde{A}_{12}, \tilde{\tilde{\varphi}}_{12}, \tilde{\tilde{B}}_{12}\big)$.

We begin by constructing a scalar $\tilde{\tilde{\varphi}}_{12}$ and require it to transform like a total derivative. Using the results of section 2.1 we have

$$\tilde{\tilde{\varphi}}_{12} = \varphi_1 \tilde{\tilde{\varphi}}_2 + \varphi_2 \tilde{\tilde{\varphi}}_1 + \frac{1}{4} B_1{}^{mn} \tilde{\tilde{B}}_{2mn} + \frac{1}{4} B_2{}^{mn} \tilde{\tilde{B}}_{1mn} - \tilde{A}_{1m} \tilde{A}_2{}^m,$$

that transforms like a total derivative. This scalar turns out to be the only one we can construct (see next subsection). The transformation law (2.5) $\delta_\varepsilon \tilde{\tilde{\varphi}}_{12} = \varepsilon^m \partial_m \varphi_{12}$ gives (up to a constant)

$$\varphi_{12} = \varphi_1 \varphi_2 + \frac{1}{4} B_{1mn} B_2{}^{mn}.$$

Using $\delta_\varepsilon \varphi_{12} = \varepsilon^m \tilde{A}_{12m}$, one gets

$$\tilde{A}_{12m} = \tilde{A}_{1m} \varphi_2 + \tilde{A}_{2m} \varphi_1 + \tilde{A}_1{}^n B_{2nm} + \tilde{A}_2{}^n B_{1nm}.$$

Next, applying the transformations (2.5) on the R.H.S. of the equation above, one finds

$$\begin{aligned}
\delta_\varepsilon \tilde{A}_{12m} &= \varepsilon_m \left(\tilde{\tilde{\varphi}}_1 \varphi_2 + \tilde{\tilde{\varphi}}_2 \varphi_1 + 2 \tilde{A}_{1m} \tilde{A}_2{}^m \right) \\
&+ \varepsilon^n \left(\tilde{\tilde{B}}_{2mn} \varphi_1 + B_{2mn} \tilde{\tilde{\varphi}}_1 + \tilde{\tilde{B}}_{1mn} \varphi_2 + B_{1mn} \tilde{\tilde{\varphi}}_2 + \tilde{\tilde{B}}_{1pn} B_2{}^p{}_m + \tilde{\tilde{B}}_{2pn} B_1{}^p{}_m \right).
\end{aligned}$$

But, in order to have a 3SUSY multiplet, we should have $\delta_\varepsilon \tilde{A}_{12m} = \varepsilon_m \tilde{\tilde{\varphi}}_{12} + \varepsilon^n \tilde{\tilde{B}}_{12mn}$, with $\tilde{\tilde{B}}_{12mn}$ a self-dual two-form, which is clearly impossible. Thus we cannot build in such a way a third Ξ_{++} multiplet starting from two Ξ_{++} multiplets.

Similar calculations can be done for all possible triplets of multiplets, leading to the same result.

Thus, we see that the simplest idea does not work. Next, in the following two subsections, we will try to construct, in all generality, invariant interacting terms. For this, we proceed in two main steps.

Firstly, we start with a given multiplet of the type $\Xi_{\pm\pm}$. Then, we find the possible sets of fields Ψ (content and transformation laws) which couple to $\Xi_{\pm\pm}$ in an invariant way.

Secondly, having obtained Ψ, we would like to get it as a function $F(\Xi_{++}, \Xi_{--}, \Xi_{+-}, \Xi_{-+})$. This function will be proven to be linear in the fields. Hence, one can get at most quadratic terms, and therefore, no interactions are possible.

3.3 Possible couplings of a given multiplet

In this subsection we couple a given 3SUSY multiplet $\Xi_{\pm\pm}$ with a set of fields Ψ. Let this multiplet be of the type $\Xi_{++} = \big(\varphi, B_+, \tilde{A}, \tilde{\tilde{\varphi}}, \tilde{\tilde{B}}_+\big)$ with $\varphi, \tilde{\tilde{\varphi}}$ two scalars, $B_+, \tilde{\tilde{B}}_+$ two self-dual 2−forms and $\tilde{A}$ a vector. The other cases $(\Xi_{--}, \Xi_{+-}, \Xi_{-+})$ are treated along the same lines.

The most general possibility of coupling, in a quadratic way, Ξ_{++} with the new fields is

$$\mathcal{L}(\Xi_{++}, \Psi) = \varphi\tilde{\tilde{\psi}} + \tilde{\tilde{\varphi}}\psi + \frac{1}{4}B_+{}^{mn}\tilde{\tilde{\psi}}_{mn} + \frac{1}{4}\tilde{\tilde{B}}_{+mn}\psi^{mn} - \tilde{A}_m\tilde{\psi}^m \tag{3.11}$$

with $\psi, \tilde{\tilde{\psi}}$ two scalars, $\psi_{mn}, \tilde{\tilde{\psi}}_{mn}$ two self-dual 2−forms (since $B_{-mn}B_+^{mn} = 0$, if B_{-mn} is anti-self-dual) and $\tilde{\psi}_m$ a vector. In (3.11) *a priori* some of the ψ fields could be set to zero. Also, the ψ fields can or cannot contain derivative terms. We treat these cases separately.

> **II:** *If the ψ fields contain no derivative terms and (3.11) is invariant, then they form a multiplet of the type Ξ_{++}.*

After an easy calculation, one gets

$$\begin{aligned}
\delta_\varepsilon\mathcal{L}(\Xi_{++}, \Psi) &= \varphi\Big(\delta_\varepsilon\tilde{\tilde{\psi}} - \varepsilon^m\partial_m\psi\Big) + \tilde{\tilde{\varphi}}\Big(\delta_\varepsilon\psi - \varepsilon_m\tilde{\psi}^m\Big) + \frac{1}{4}B_{+mn}\Big(\delta_\varepsilon\tilde{\tilde{\psi}}^{mn} - \varepsilon^p\partial_p\psi^{mn}\Big) \\
&+ \frac{1}{4}\tilde{\tilde{B}}_{+mn}\Big(\delta_\varepsilon\psi^{mn} + \varepsilon^m\tilde{\psi}^n - \varepsilon^n\tilde{\psi}^m - i\varepsilon^{mnpq}\varepsilon_p\tilde{\psi}_q\Big) \\
&- \tilde{A}_m\Big(\delta_\varepsilon\tilde{\psi}^m - \varepsilon_n\tilde{\tilde{\psi}}^{mn} - \varepsilon^m\tilde{\tilde{\psi}}\Big) + \varepsilon^p\partial_p\Big(\varphi\psi + \frac{1}{4}B_{+mn}\psi^{mn}\Big).
\end{aligned} \tag{3.12}$$

By hypothesis, all the fields appearing in $\delta_\varepsilon\mathcal{L}(\Xi_{++}, \Psi)$ do not contain derivative terms. This means that no integration by part can be done, thus no more total derivatives can be present. The invariance of $\mathcal{L}(\Xi_{++}, \Psi)$ subsequently means that $\delta_\varepsilon\mathcal{L}(\Xi_{++}, \Psi) = 0$ and the ψ fields transform as a Ξ_{++} multiplet (see (2.5)). This means that all the fields ψ are present in (3.11).

In the previous case, we did not consider any derivative terms. Now, if we assume that the ψ fields contain only first derivative terms, their most general form is

$$\begin{aligned}
&\psi = \partial_m\lambda^m, \quad \tilde{\tilde{\psi}} = \partial_m\tilde{\tilde{\lambda}}^m \\
&\psi_{mn} = \partial_m\lambda'_n - \partial_n\lambda'_m - i\varepsilon_{mnpq}\partial^p\lambda'^q \\
&\tilde{\tilde{\psi}}_{mn} = \partial_m\tilde{\tilde{\lambda}}'_n - \partial_n\tilde{\tilde{\lambda}}'_m - i\varepsilon_{mnpq}\partial^p\tilde{\tilde{\lambda}}'^q, \\
&\tilde{\psi}_m = \partial_m\tilde{\lambda} + \partial^n\tilde{\lambda}_{nm}.
\end{aligned} \tag{3.13}$$

(with $\psi_{mn}, \tilde{\tilde{\psi}}_{mn}$ self-dual 2−forms) with $\tilde{\lambda}$ a scalar, $\lambda_m, \lambda'_m, \tilde{\tilde{\lambda}}_m, \tilde{\tilde{\lambda}}'_m$ four vectors and λ_{mn} a 2−form (whose anti-(self-)dual character is not specified at that point).

> **III:** *If the ψ fields are as in (3.13) and the Lagrangian (3.11) is invariant, then they form a Ξ_{++} multiplet.*

As before the variation of (3.11) gives (3.12). It is more natural to obtain the variations of the fields ψ instead of the ones of the fields λ. Since now we have allowed derivative couplings, some integration by part can be done leading to total derivatives. This means, in particular, that *a priori* one cannot put $\delta_\varepsilon\mathcal{L}(\Xi_{++},\Psi)=0$. Paying attention to this possibility one gets

$$\begin{aligned}
&\delta_\varepsilon\psi=\varepsilon^m\tilde{\psi}_m,\quad \delta_\varepsilon\tilde{\tilde{\psi}}=\varepsilon^m\partial_m\psi,\quad \delta_\varepsilon\tilde{\psi}^m=\varepsilon^m\tilde{\tilde{\psi}}+\varepsilon_n\tilde{\tilde{\psi}}^{mn}\\
&\delta_\varepsilon\psi_{mn}=\varepsilon_n\tilde{\psi}_m-\varepsilon_m\tilde{\psi}_n+i\varepsilon_{mnpq}e^p\tilde{\psi}^q\\
&\delta_\varepsilon\tilde{\tilde{\psi}}_{mn}=\varepsilon^r\partial_r\psi_{mn}
\end{aligned}\tag{3.14}$$

Indeed, as in the previous case, no other total derivative can appear. For instance, looking at the variation of ψ_{mn} one could have $\delta_\varepsilon\psi_{mn}=\varepsilon_n\tilde{\psi}_m-\varepsilon_m\tilde{\psi}_n+i\varepsilon_{mnpq}e^p\tilde{\psi}^q+X_{mn}$ with $B^{mn}X_{mn}$ being a total derivative. Taking into account the various possibilities to built such an X_{mn} from the λ fields, it is not difficult to check that $X_{mn}=0$. Finally, the transformation laws of the λ fields can now be deduced form (3.14). For instance, one gets

$$\delta_\varepsilon\tilde{\tilde{\lambda}}_m=\varepsilon^p\partial_p\lambda_m+a(\varepsilon_m\partial_n\lambda^n-\varepsilon_n\partial^n\lambda_m)+a'(\varepsilon_m\partial_n\lambda'^n-\varepsilon_n\partial^n\lambda'_m),\tag{3.15}$$

where a,a' are arbitrary constants. Similarly one can obtain the variations of the other λ fields, but it is not necessary for our purpose.

Note that the coupling Lagrangian $\mathcal{L}_c(\Xi_{++},\Xi_{+-})$ in (2.18) is a special case of the above study where $\lambda'_m\equiv\lambda_m$, $\tilde{\tilde{\lambda}}'_m\equiv\tilde{\tilde{\lambda}}_m$ in eq.(3.13). In this case, the ψ's form the derivative multiplet, (3.6), of the λ's.

We treat now the most general case, when

$$\begin{aligned}
&\psi=\rho+\partial_m\lambda^m,\quad \tilde{\tilde{\psi}}=\tilde{\tilde{\rho}}+\partial_m\tilde{\tilde{\lambda}}^m,\\
&\psi_{mn}=\rho_{mn}+\partial_m\lambda'_n-\partial_n\lambda'_m-i\varepsilon_{mnpq}\partial^p\lambda'^q,\\
&\tilde{\tilde{\psi}}_{mn}=\tilde{\tilde{\rho}}_{mn}+\partial_m\tilde{\tilde{\lambda}}'_n-\partial_n\tilde{\tilde{\lambda}}'_m-i\varepsilon_{mnpq}\partial^p\tilde{\tilde{\lambda}}'^q,\\
&\tilde{\psi}_m=\tilde{\rho}_m+\partial_m\tilde{\lambda}+\partial^n\tilde{\lambda}_{nm}.
\end{aligned}\tag{3.16}$$

> **IV:** *If the ψ are as in (3.16) and the Lagrangian (3.11) is invariant, then they transform as in (3.14).*

The proof is analogous to the case **III**. Here again the ψ fields must all be present in (3.11), but now some of the λ or ρ fields can be absent in (3.16).

In **II**, **III** or **IV** we have assumed that the fields ψ contain at most one derivative. One should of course address the more general case where higher number of derivatives are allowed. In fact, in this case also, the results remain unchanged.

If one considers terms with two derivatives, the only scalar, vector and 2-form that can built starting from a scalar λ, a vector λ_m and a 2-form λ_{mn}, are $\psi = \Box\lambda + \partial^m\lambda_m$, $\psi_m = \Box\psi_m + a\partial_m\partial^n\psi_n$ and $\psi_{mn} = \Box\lambda_{mn} + b\partial_{[m}\partial^p\lambda_{n]+p}$. After introducing the fields $\Psi = (\psi, \psi_{mn}, \tilde{\psi}_m, \tilde{\tilde{\psi}}, \tilde{\tilde{\psi}}_{mn})$, with two derivatives as above, the invariance of (3.11) requires, as in the proof of property **II**, that Ψ is a Ξ_{++} multiplet. When the fields are expressed with pure d'Alembertian, we obtain nothing else but the Lagrangian of the type $\mathcal{L}_0$ in (2.6). If we reiterate the process with an even number of derivatives, similar arguments lead to the same conclusion that property **II** holds. For the case of an odd number of derivatives, property **III** holds. The two cases are different, because of the possibility of using identity (3.9) for an odd number of derivatives.

As previously stated similar results hold when one studies the Ξ_{--}, Ξ_{+-} and Ξ_{-+} multiplets.

In this subsection, we have shown that the invariance of (3.11) implies some specific behaviour of the fields ψ. The next step is to construct these ψ fields from the multiplets $\Xi_{\pm\pm}$.

3.4 Generalised tensor calculus

The purpose of this subsection, is to explicitly get, out of all the 3SUSY multiplets, the fields ψ found in the previous subsection. In other words, we want to find functions

$$F(\Xi_{++}, \Xi_{--}, \Xi_{+-}, \Xi_{-+}) = \Big(f, f_{mn}, \tilde{f}_m, \tilde{\tilde{f}}, \tilde{\tilde{f}}_{mn}\Big), \tag{3.17}$$

where $f, f_{mn}, \tilde{f}_m, \tilde{\tilde{f}}, \tilde{\tilde{f}}_{mn}$ depend on the four multiplets, and transform like in the previous subsection. Of course when one writes that a function depends on a multiplet $\Xi_{\pm\pm}$ this means that it depends on its fields as well as on the derivatives of its fields.

In this subsection also we concentrate on the Ξ_{++} multiplet, since the other cases are similar.

V: *The only function F defined as in (3.17), with at most first order derivatives in the fields and transforming as a Ξ_{++} multiplet is*

$$F(\Xi_{++}, \Xi_{--}, \Xi_{+-}, \Xi_{-+}) = \alpha\Xi_{++} + \beta\mathcal{D}\Xi_{+-}, \alpha, \beta \in \mathbb{C}.$$

We first consider the case where no derivative dependence in the fields is present in the functions f. Furthermore, we assume in a first time that F depends only on the first multiplet Ξ_{++}. We start by writing the transformation laws of the f fields, like *e.g.*

$$\delta_\varepsilon \tilde{\tilde{f}} = \varepsilon^m\partial_m f = \frac{\partial\tilde{\tilde{f}}}{\partial\tilde{\tilde{\varphi}}}\delta_\varepsilon\tilde{\tilde{\varphi}} + \frac{\partial\tilde{\tilde{f}}}{\partial\varphi}\delta_\varepsilon\varphi + \frac{1}{2}\frac{\partial\tilde{\tilde{f}}}{\partial B_{mn}}\delta_\varepsilon B_{mn} + \frac{1}{2}\frac{\partial\tilde{\tilde{f}}}{\partial\tilde{\tilde{B}}_{mn}}\delta_\varepsilon\tilde{\tilde{B}}_{mn} + \frac{\partial\tilde{\tilde{f}}}{\partial\tilde{A}_m}\delta_\varepsilon\tilde{A}_m. \tag{3.18}$$

Substituting in (3.18) the variations (2.5) of the fields we get

$$\begin{aligned}\varepsilon^m\partial_m f &= \frac{\partial\tilde{\tilde{f}}}{\partial\tilde{\tilde{\varphi}}}\varepsilon^m\partial_m\varphi + \frac{\partial\tilde{\tilde{f}}}{\partial\varphi}\varepsilon^m\tilde{A}_m - 2\frac{\partial\tilde{\tilde{f}}}{\partial B_{mn}}\varepsilon_m\tilde{A}_n \\ &+ \frac{1}{2}\frac{\partial\tilde{\tilde{f}}}{\partial\tilde{\tilde{B}}_{mn}}\varepsilon^p\partial_p B_{mn} + \frac{\partial\tilde{\tilde{f}}}{\partial\tilde{A}_m}(\varepsilon_m\tilde{\tilde{\varphi}} + \varepsilon^n\tilde{\tilde{B}}_{mn}).\end{aligned} \tag{3.19}$$

In the L.H.S. of the equation above we have only one derivative. In the R.H.S. since $\tilde{\tilde{f}}$ depends only on the fields and not on its derivatives, we have

$$\frac{\partial\tilde{\tilde{f}}}{\partial\varphi} = 0, \frac{\partial\tilde{\tilde{f}}}{\partial B_{mn}} = 0, \frac{\partial\tilde{\tilde{f}}}{\partial\tilde{A}_m} = 0.$$

Then, after integration by parts (3.18) reduces to

$$\partial_m(\tilde{\tilde{f}} - \frac{\partial\tilde{\tilde{f}}}{\partial\tilde{\tilde{\varphi}}}\varphi - \frac{1}{2}\frac{\partial\tilde{\tilde{f}}}{\partial\tilde{\tilde{B}}_{mn}}B_{mn}) = -\varphi\partial_m\frac{\partial\tilde{\tilde{f}}}{\partial\tilde{\tilde{\varphi}}} - \frac{1}{2}B_{np}\partial_m\frac{\partial\tilde{\tilde{f}}}{\partial\tilde{\tilde{B}}_{np}}.$$

Arguing as before, one gets $\partial_m\frac{\partial\tilde{\tilde{f}}}{\partial\tilde{\tilde{\varphi}}} = 0, \partial_m\frac{\partial\tilde{\tilde{f}}}{\partial\tilde{\tilde{B}}_{np}} = 0$, and

$$\tilde{\tilde{f}} = \alpha\tilde{\tilde{\varphi}} + \frac{1}{2}X_{mn}\tilde{\tilde{B}}^{mn}.$$

The transformation law of $\tilde{\tilde{f}}$ easily gives $f = \alpha\varphi + \frac{1}{2}X_{mn}B^{mn}$. Then, the transformation law of f gives $\tilde{f}_m = \alpha\tilde{A}_m - 2X_{mn}\tilde{A}^n$. Finally, the transformation law of $\tilde{f}_m$ gives $X_{mn} = 0$, and $\tilde{\tilde{f}}_{mn} = \alpha\tilde{\tilde{B}}_{mn}$. Finally we get

$$F(\Xi_{++}) = \alpha\Xi_{++}.$$

Now, when one takes into account that F depends on all the multiplets, $F(\Xi_{++}, \Xi_{+-}, \Xi_{+-}, \Xi_{--})$, one obtains, by arguments along the same lines

$$F(\Xi_{++}, \Xi_{+-}, \Xi_{+-}, \Xi_{--}) = \alpha\Xi_{++}.$$

Now we treat the general case, namely when F depends on the all the multiplets (2.3) as well as on their first order derivatives. In this case the proof is more intricate since terms like $\frac{\partial f}{\partial\partial_m\phi}\delta_\varepsilon\partial_m\varphi$ are present and

$$F(\Xi_{++}, \Xi_{+-}, \Xi_{+-}, \Xi_{--}) = \alpha\Xi_{++} + \beta\mathcal{D}\Xi_{+-}. \tag{3.20}$$

with $\mathcal{D}\Xi_{+-}$ being defined in (3.6).

So far we have considered the possibility to build only Ξ_{++} multiplets. One could address the possibility to obtain the fields λ transforming as (3.15). Assume now, that one can non-linearly built such λ which we denote generically by $\Lambda = G(\Xi)$. From (3.13) written generically as $\Psi = \partial\Lambda$, one can non-linearly obtain $\Psi = \partial G(\Xi) = F(\Xi, \mathcal{D}\Xi)$. Since the ψ

fields Ψ form a Ξ_{++} multiplet (see (3.14)) such a non-linear function does not exists (see **V**). This means that there is no non-linear functions leading to the λ fields.

The case of functions involving higher number of derivatives goes along the same lines. This leads to the new possibilities $F(\Xi_{++}, \Xi_{+-}, \Xi_{+-}, \Xi_{--}) = \alpha \Box^n \Xi_{++}$ (resp. $F(\Xi_{++}, \Xi_{+-}, \Xi_{+-}, \Xi_{--}) = \alpha \Box^n \mathcal{D} \Xi_{+-}$ for an even (resp. odd) numbers of derivatives.

4 Compatibility with $U(1)$ gauge symmetry

In this section we address the question of compatibility between the 3SUSY and the $U(1)$ gauge transformations. We note first that the Lagrangians given in Eqs. (2.6), (2.18) are invariant under the following transformations

$$\begin{aligned}
\phi &\rightarrow \phi + k, \\
\mathcal{A}_m &\rightarrow \mathcal{A}_m + \partial_m \chi, \\
\mathcal{B}^{(\pm)}_{mn} &\rightarrow \mathcal{B}^{(\pm)}_{mn} + \partial_m \chi_n - \partial_n \chi_m \mp i \varepsilon_{mnpq} \partial^p \chi^q
\end{aligned} \tag{4.1}$$

with

$$\partial_m k = 0, \quad \Box \chi = 0, \quad \Box \chi_n - \partial_n \partial_m \chi^m = 0 \tag{4.2}$$

where $\phi(x), \mathcal{A}_m(x), \mathcal{B}^{(\pm)}_{mn}(x)$ denote generically the $0-$, $1-$ and $2-$form fields appearing in the various Ξ multiplets and $(\pm)$ indicates the self-duality properties (3.1). The transformations in (4.1) can be qualified as gauge transformations. Indeed, strictly speaking, the gauge transformations should be required for the *real-valued* fields defined in (2.7). However, for the $0-$ and $1-$forms these have the same form as those in (4.1) by linearity. The case of the $2-$forms is somewhat different: the usual gauge transformations of the real-valued fields (2.7) which read

$$\begin{aligned}
B_{1mn} &\rightarrow B_{1mn} + \partial_m \chi_n - \partial_n \chi_m \\
\tilde{\tilde{B}}_{1mn} &\rightarrow \tilde{\tilde{B}}_{1mn} + \partial_m \tilde{\tilde{\chi}}_n - \partial_n \tilde{\tilde{\chi}}_m
\end{aligned} \tag{4.3}$$

lead to the transformations (4.1) for $\mathcal{B}^{(\pm)}$ as a consequence of projecting out the (anti)-self-duality content of the real-valued $2-$forms (see (2.7), (2.8)). Nonetheless, for arbitrary k, χ and χ_m, the transformations (4.1) do not preserve in general the 3SUSY multiplet structures, so that the 3SUSY invariance of the gauge transformed Lagrangian loses its meaning. Namely, it seems difficult to put in the same 3SUSY multiplet the gauge parameters $((k, \chi_m, \tilde{\chi}, \tilde{\tilde{k}}, \tilde{\tilde{\chi}}_m)$ for say the Ξ_{++} multiplet). It is thus mandatory, for the sake of consistence, to seek for subclasses of gauge transformations of the form $\delta_{gauge} \Xi = \Lambda$ where Ξ and Λ are 3SUSY multiplets of the same type. [Recall that in the case of usual supersymmetry, this is achieved rather transparently in terms of superfields in the form $V \rightarrow V + \Phi + \Phi^\dagger$ [14].] In the present case, not having a superfield formulation at our disposal, we will make

use of the derivative multiplets defined in section **3.1**.

Before studying further this point, a general remark is in order here: the Lagrangians (2.6), (2.18) are also invariant under a general shift transformation $\Xi \to \Xi \pm \Theta_0$ provided that Ξ and $\Theta_0 = (..., \theta_0, ..., \theta_0^m, ...\theta_0^{mn}, ...)$ are of the same type and that the components of Θ_0 satisfy conditions similar to (4.2), $\partial_m \Theta_0 \equiv (..., \partial_m \theta_0, ..., \partial_m \theta_0^m, ...\partial_m \theta_0^{mn}, ...) = 0$ together with $\Box \theta_0^m = 0$, but where its $1-$ and $2-$form components do not necessarily have to be differential *exact* forms as required by a gauge transformation[4]. Furthermore, combining this transformation with a 3SUSY transformation, one finds an invariance under $\Xi \to \Xi + \delta_\varepsilon \Theta_0$ as a consequence of the following series of equalities (up to surface terms):

$$\begin{aligned} &\mathcal{L}(\Xi) \hat{=} \mathcal{L}(\Xi + \delta_{-\varepsilon}\Xi) \hat{=} \mathcal{L}(\Xi + \delta_{-\varepsilon}\Xi + \Theta_0) \hat{=} \\ &\mathcal{L}(\Xi + \Theta_0 + \delta_{-\varepsilon}(\Xi + \Theta_0) + \delta_\varepsilon \Theta_0) \hat{=} \mathcal{L}(\Xi + \Theta_0 + \delta_\varepsilon \Theta_0) \hat{=} \mathcal{L}(\Xi + \delta_\varepsilon \Theta_0). \end{aligned}$$

It is worth noting that the condition on $\partial_m \Theta_0$ is not preserved by 3SUSY, as can be seen from the $0-$ and $2-$form transformations (2.5)- that is $\partial_m \delta_\varepsilon \Theta_0 \neq 0$. Thus, the transformation

$$\delta \Xi \equiv \delta_\varepsilon \Theta_0 \tag{4.4}$$

identified above provides indeed a new symmetry.

Let us now consider the specific case of gauge transformations. Since we need simultaneously derivatives and definite 3SUSY multiplet structures, one can make use of eq.(3.10) and seek for a transformation of the form

$$\Xi_{s\pm} \to \Xi_{s\pm} + \mathcal{D}\Lambda_{s\mp} \ , \qquad (s = +, -) \tag{4.5}$$

For instance starting from $\Xi_{++} = \left(\varphi, B_{mn}, \tilde{A}_m, \tilde{\tilde{\varphi}}, \tilde{\tilde{B}}_{mn}\right)$ and $\Lambda_{+-} = \left(\lambda_m, \tilde{\lambda}, \tilde{\lambda}_{mn}, \tilde{\tilde{\lambda}}_m\right)$ one has, (3.6),

$$\begin{array}{rcl} \varphi & \to & \varphi + \partial_m \lambda^m \\ B_{mn} & \to & B_{mn} + \partial_m \lambda_n - \partial_n \lambda_m - i\varepsilon_{mnpq}\partial^p \lambda^q \\ \tilde{A}_m & \to & \tilde{A}_m + \partial_m \tilde{\lambda} + \partial^n \tilde{\lambda}_{nm} \\ \tilde{\tilde{\varphi}} & \to & \tilde{\tilde{\varphi}} + \partial^m \tilde{\tilde{\lambda}}_m \\ \tilde{\tilde{B}}_{mn} & \to & \tilde{\tilde{B}}_{mn} + \partial_m \tilde{\tilde{\lambda}}_n - \partial_n \tilde{\tilde{\lambda}}_m - i\varepsilon_{mnpq}\partial^p \tilde{\tilde{\lambda}}^q \end{array} \tag{4.6}$$

The invariance conditions (4.2) read here $\partial_m \mathcal{D}\Lambda = 0$ and reduce to (see also footnote 4),

$$\partial_m \left(\partial \cdot \lambda\right) = \partial_m \left(\partial \cdot \tilde{\tilde{\lambda}}\right) = 0 \tag{4.7}$$

$$\Box \lambda_m = \Box \tilde{\tilde{\lambda}}_m = 0 \tag{4.8}$$

$$\Box \tilde{\lambda} + \partial^m \partial^n \tilde{\lambda}_{nm} \equiv \Box \tilde{\lambda} = 0 \tag{4.9}$$

[4]We stress that the similarity with (4.2) is a consequence of the (anti)-self-duality of θ_0^{mn}. Indeed, while the field strengths H_{mnp} (and their duals) appearing in (2.6), are *automatically* invariant under the transformation of $\mathcal{B}$ in (4.1), their invariance under this new transformation (where θ_0^{mn} is (anti)-self-dual), requires $\partial_m \theta_0^{mn} = 0$ as can be shown by using the first identity in (3.4).

One further constraint comes from the requirement that the transformation of $\tilde{A}$ in (4.6) should be an exact form as in the gauge transformations (4.1), that is

$$\partial^n \tilde{\lambda}_{nm} \equiv \partial_m \chi \tag{4.10}$$

This constraint is not trivial, and for $\tilde{\lambda}_{nm}$ anti-self-dual it implies

$$\Box \tilde{\lambda}_{nm} = 0 \tag{4.11}$$

as can be proven by using the first equation in (3.4)[5]. Furthermore, the anti-symmetry of $\tilde{\lambda}_{nm}$ leads trivially to

$$\Box \chi = 0. \tag{4.12}$$

To proceed, our strategy will be as follows: determine first the general functional forms of $\lambda_m, \tilde{\tilde{\lambda}}_m$ satisfying the constraints (4.7, 4.8), and those of $\tilde{\lambda}, \chi$ satisfying (4.9, 4.12). Then, knowing χ, construct explicitly a general antisymmetric, anti-self-dual 2−form satisfying (4.10), which would then automatically satisfy (4.11). The fact that the components of Λ_{+-} can indeed be *general functions* (in terms of variables yet to be identified), despite the gauge fixing conditions (4.7 -4.9, 4.11), is crucial to assess the elimination of unphysical degrees of freedom of the fields Ξ. It is worth stressing that these gauge fixing conditions do not allow to choose in general Lorentz gauges, $\partial_m \tilde{A}^m = 0$, or $\partial^n B_{nm} = 0$. Other gauges such as the Coulomb gauge, the axial gauge, etc... can in principle be imposed. However, if for instance the scalar functions $\tilde{\lambda}$ and χ depend only on the space-time Lorentz invariant $x_m x^m$, then the conditions (4.9, 4.12) determine uniquely their functional form, $\tilde{\lambda}[x^2] \sim \chi[x^2] \sim 1/x^2$ up to some additive constants. In this case only special space-time configurations of the field $\tilde{A}_m$ can be eliminated by a gauge choice. It is thus tempting to consider more general trial functions $\mathcal{F}[x^2,\ \xi \cdot x,\ A_{mn} x^m x^n, \cdots]$, where ξ_m and A_{mn} are some constant 4−vector and symmetric tensor. The inclusion of a 4−vector ξ_m is somewhat natural in the context of the 3SUSY algebra whose generators (and transformation parameters) are also 4−vectors. On the other hand, one can also include a dependence on constant (anti)-self-dual 2−forms R_{mn} which sit in the same 3SUSY multiplet as ξ_m. For instance $A_{mn} \equiv R_{mp} R^p{}_n$ induces a dependence on R^2 in $\mathcal{F}$ (see eq.(3.2)), while a dependence on invariants such as $R_{mp} x^m \xi^p$ will turn out to be also natural to consider.

To start with, we studied functions of x^2 and $\xi \cdot x$ and established the following properties.

(i): $\Box \mathcal{F}[x^2,\ \xi \cdot x] = 0$ *has generic solutions if and only if* $\xi^2 = 0$*; these solutions take the form:*

$$\mathcal{F}(x^2,\ \xi \cdot x) = G[\xi \cdot x] + (\xi \cdot x)^{-1} H[\frac{x^2}{(\xi \cdot x)}]$$

where G *and* H *are arbitrary functions.*

[5] Obviously, the various properties discussed here as well as in section **3.1** could be also derived compactly in the language of differential geometry.

(ii): *if $\xi^2 \neq 0$, the general solution for $\Box \mathcal{F}[x^2,\ \xi \cdot x] = 0$ takes the more particular form*

$$\mathcal{F}[x^2,\ \xi \cdot x] = ((\xi \cdot x)^2 - \frac{\xi^2 x^2}{4})(\frac{C_1}{(x^2)^3} + C_2) + C_3 \frac{(\xi \cdot x)}{(x^2)^2} + C_4\ \xi \cdot x + \frac{C_5}{x^2} + C_6$$

where the C_i are arbitrary constants.

The above properties determine the general form of the fields $\tilde{\lambda}$ and χ subject to (4.9, 4.12). In the sequel we will stick to the case $\xi^2 = 0$ since, according to **(i)**, it allows the most general configurations for the gauge transformations. The general form for the fields $\lambda_m, \tilde{\tilde{\lambda}}_m$ is then determined by the following property which we established,

(iii): *The general solution for a $1-$form $\mathcal{F}_m[x^p, \xi^p]$ (with $\xi^2 = 0$), subject to the two constraints, $\Box \mathcal{F}_m = 0$ and $\partial_m(\partial \cdot \mathcal{F}) = 0$, is*

$$\mathcal{F}_m = g[\xi \cdot x]\xi_m + \alpha x_m + (\frac{1}{(x^2)^2}\alpha_{mr} + \beta_{mr})x^r + \kappa(\frac{x^2}{(\xi \cdot x)^3}\xi_m - \frac{x_m}{(\xi \cdot x)^2})$$

where g is an arbitrary function, $\kappa, \alpha, \beta_{mn}$ arbitrary constants and α_{mn} an arbitrary anti-symmetric tensor.

Thus,

$$\tilde{\lambda}[\xi \cdot x, x^2] = G_1[\xi \cdot x] + (\xi \cdot x)^{-1} H_1[\frac{x^2}{(\xi \cdot x)}] \tag{4.13}$$

$$\chi[\xi \cdot x, x^2] = G_2[\xi \cdot x] + (\xi \cdot x)^{-1} H_3[\frac{x^2}{(\xi \cdot x)}] \tag{4.14}$$

$$\begin{aligned} \lambda_m[\xi \cdot x, x^2] \quad = \quad & g_1[\xi \cdot x]\xi_m + \alpha x_m + (\frac{1}{(x^2)^2}\alpha_{mr} + \beta_{mr})x^r \\ & + \kappa_1(\frac{x^2}{(\xi \cdot x)^3}\xi_m - \frac{x_m}{(\xi \cdot x)^2}) \end{aligned} \tag{4.15}$$

$$\begin{aligned} \tilde{\tilde{\lambda}}_m[\xi \cdot x, x^2] \quad = \quad & \tilde{\tilde{g}}_1[\xi \cdot x]\xi_m + \tilde{\tilde{\alpha}} x_m + (\frac{1}{(x^2)^2}\tilde{\tilde{\alpha}}_{mr} + \tilde{\tilde{\beta}}_{mr})x^r \\ & + \tilde{\tilde{\kappa}}_1(\frac{x^2}{(\xi \cdot x)^3}\xi_m - \frac{x_m}{(\xi \cdot x)^2}) \end{aligned} \tag{4.16}$$

Starting from (4.14), one can construct explicitly $\tilde{\lambda}_{mn}$ satisfying the constraint (4.10), in the form

$$\tilde{\lambda}_{mn}[\xi \cdot x, x^2] = x_{[m}\xi_{n]_-} F[\xi \cdot x, x^2] \tag{4.17}$$

The function F can be determined in terms of G_2, H_3 appearing in (4.14). One finds

$$F[\xi\cdot x, x^2] = -(\xi\cdot x)^{-2}H_3[\frac{x^2}{(\xi\cdot x)}] + (\xi\cdot x)^{-1}G_2[\xi\cdot x] - 2(\xi\cdot x)^{-3}\int_0^{\xi\cdot x} G_2[t]\,t\,dt \qquad (4.18)$$

To summarize, we have proven the existence of gauge transformations which preserve the type of 3SUSY multiplets and satisfy the necessary constraints for gauge invariance. However, the gauge transformation functions are found to be not completely arbitrary. As can be seen from eqs.(4.13, 4.14) arbitrary gauge fixing can be *a priori* applied to arbitrary field configurations of $\tilde{A}_m$ as a function of $\xi\cdot x$, while configurations in x^2 can be gauge-fixed only in conjunction with $\xi\cdot x$. In contrast, gauge transformations of B_{mn} and $\tilde{\tilde{B}}_{mn}$ correspond only to specific functions of x^2. Indeed, direct inspection of (4.15, 4.16) shows that $\partial_{[m}\lambda_{n]_-}, (\partial_{[m}\tilde{\tilde{\lambda}}_{n]_-})$ appearing in (4.6), receive contributions only from $\alpha_{mr}, \beta_{mr}, (\tilde{\tilde{\alpha}}_{mr}, \tilde{\tilde{\beta}}_{mr})$. To eliminate unphysical degrees of freedom for more general field configurations, one can make use of the symmetry (4.4). However, one should keep in mind that even for this non-gauge transformation, the required constraints $\partial_m\Theta_0 = 0$ and $\Box\theta_0^m = 0$ will somewhat reduce the generality of the field configurations.

Hence, it seems that one is lead, at this stage of the analysis, to the unusual feature that the number of physical degrees of freedom of the gauge fields depends on the space-time configurations of these fields! However a more thorough study is still needed and is actually akin to the way space-time itself transforms under 3SUSY. The fact that x^m should transform non-trivially under 3SUSY is obvious from the presence of ∂_m in (2.5) which implies that the transformation law of a field of gradation 1 depends on the space-time configuration of its partner field of gradation -1 [e.g. $\tilde{\varphi}$ does not transform if φ is constant in x^m, etc...]. A related question, not addressed so far, is whether the $0-$, $1-$, and $2-$form fields should verify some space-time constraints in order to be members of the same 3SUSY multiplet. The answer to such a question depends crucially on the way x^m transforms under 3SUSY, which in turn depends on the possibility to define a space of parameters including x^m and leading to the correct transformations of the fields. We give here for illustration one example of how a non-trivial transformation of x_m can induce constraints. Assume that x^m belongs to a $(+,-)$ multiplet $X_{+-} = (x_m, \tilde{\alpha}, \tilde{R}_{mn}, \tilde{\tilde{\xi}}_m)$ where $\tilde{\alpha}, \tilde{R}_{mn}, \tilde{\tilde{\xi}}_m$ are x independent (x_m being of gradation 0, $\tilde{\alpha}, \tilde{R}_{mn}$ of gradation 1 and $\tilde{\tilde{\xi}}_m$ of gradation 2). One then has $\delta_\varepsilon x_m = \varepsilon^n\tilde{R}_{mn} + \varepsilon_m\tilde{\alpha}$, $\delta_\varepsilon\tilde{\alpha} = \varepsilon\cdot\tilde{\tilde{\xi}}, \delta_\varepsilon\tilde{R}_{mn} = -(\varepsilon_m\tilde{\tilde{\xi}}_n - \varepsilon_n\tilde{\tilde{\xi}}_m) - i\varepsilon_{mnpq}\varepsilon^p\tilde{\tilde{\xi}}^q$, and $\delta_\varepsilon\tilde{\tilde{\xi}}_m = \varepsilon_m$. (We do not need to worry here about the fact that δ_ε does not generically preserve the reality of x^m, $\tilde{R}_{mn}$ being complex valued, as this would just add an extra constraint to the ones we are illustrating here.) Let us now try to construct from X_{+-} a $(+,+)$ multiplet $\Xi_{++} = \big(\varphi, B, \tilde{A}, \tilde{\tilde{\varphi}}, \tilde{\tilde{B}}\big)$. This turns out to be extremely constrained. For instance, starting from an arbitrary function $\tilde{\tilde{\varphi}} \equiv \tilde{\tilde{\varphi}}(\tilde{\tilde{\xi}}\cdot\tilde{\tilde{\xi}})$, one finds that the only consistent possibility requires

$$\tilde{\alpha} = \frac{(\tilde{\tilde{\xi}}\cdot\tilde{\tilde{\xi}})}{2}, \qquad (4.19)$$

and reads:

$$
\begin{aligned}
&\varphi = x \cdot \tilde{\tilde{\xi}},\ B_{mn} = constant, \\
&\tilde{A}_m = \frac{1}{\sqrt[3]{4}}\,(x_m + \tilde{\tilde{\xi}}^n \tilde{R}_{nm} + \alpha \tilde{\tilde{\xi}}_m), \\
&\tilde{\tilde{B}}_{mn} = 0,\ \tilde{\tilde{\varphi}} = \frac{1}{\sqrt[3]{2}}\,\tilde{\tilde{\xi}} \cdot \tilde{\tilde{\xi}},
\end{aligned}
\tag{4.20}
$$

where, furthermore, the 3SUSY transformation of X_{+-}, $\delta_\varepsilon X_{+-}$, induces the 3SUSY transformation of Ξ_{++} but with a specifically rescaled parameter, namely $\delta_{\sqrt[3]{4}\,\varepsilon}\ \Xi_{++}$. Equations (4.19, 4.20) are equally obtained if one starts from an arbitrary function $\varphi(x \cdot \tilde{\tilde{\xi}})$, [note that (4.19) is the only relation between α and $\tilde{\tilde{\xi}}$ which is compatible with their transformation under 3SUSY] . Also similar conclusions are reached if one starts from $\tilde{\tilde{\varphi}}(\tilde{\tilde{\eta}} \cdot \tilde{\tilde{\eta}})$ or $\varphi(x \cdot \tilde{\tilde{\eta}})$ where $\tilde{\tilde{\eta}} \equiv \tilde{\tilde{\xi}}^n \tilde{R}_{nm}$, or in the case of multi-variable functions. One even hits impossibilities in more general cases where the components of X_{+-} are assumed to be $x-$dependent. Furthermore, the use of the other bosonic multiplets or their real-valued combinations (2.7) does not improve the situation.

Such strong obstructions are an indication that x^m is not sitting in the appropriate multiplets, or equivalently, that a convenient *superspace* formulation which weakens as much as possible the constraints has not yet been identified.[6] The most natural candidate for such a superspace would make use of the fermionic 3SUSY (fundamental) multiplets [4]. In this case the superspace would be spanned by $(x^m, \theta_i, \bar{\theta}_i)$, where the θ_i's $(i = 1, 2, 3)$ are $x-$independent anticommuting variables such that $(\theta_1, \bar{\theta}_2, \theta_3)$ forms a 3SUSY fermionic multiplet which verifies ([4])

$$
\begin{aligned}
\delta_\varepsilon \theta_{1\alpha} &= \varepsilon^n \sigma_{n\alpha\dot{\alpha}}\, \bar{\theta}_2^{\dot{\alpha}} \\
\delta_\varepsilon \bar{\theta}_2^{\dot{\beta}} &= \varepsilon^n \bar{\sigma}_n^{\dot{\beta}\beta}\, \theta_{3\beta} \\
\delta_\varepsilon \theta_{3\alpha} &= 0
\end{aligned}
\tag{4.21}
$$

supplemented by the corresponding rules for $(\bar{\theta}_1, \theta_2, \bar{\theta}_3)$. From here the determination of $\delta_\varepsilon x^m$ (which obviously has to be non-linear) proceeds in a well-defined way, starting from the most general "cubic superfield" fermionic multiplets $\psi_i(x^m, \theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2, \theta_3, \bar{\theta}_3)$ with $i = 1, 2, 3$. This lies, however, out of the scope of the present paper, and will be treated elsewhere.

[6] It is instructive to keep in mind the example of conventional supersymmetry. There too, one could have similarly asked whether the various scalar, spinor or vector fields should have special space-time configurations in order to belong to the same supermultiplets. For instance, it would indeed be so, in the case of chiral supermultiplets, if an inappropriate SUSY transformation of space-time is used. In this context, the usual superspace formulation can be retrieved from the requirement that such potential constraints should be completely relaxed.

5 Summary and outlook

In this paper, we have continued the study of cubic supersymmetry initiated in [4]. Here, we focused on the bosonic multiplets leaving aside the fermionic ones. We considered the most general 3SUSY invariant Lagrangian which is quadratic in the fields, exhibited explicitly its diagonal form and argued for a possible solution for the unboundedness from below of the energy density encountered in [4]. Furthermore, we studied interaction terms involving *only* the bosonic multiplets and proved that 3SUSY forbids such terms altogether. Such an obstruction strengthens the interpretation of the boson multiplets in terms of abelian gauge fields for which "renormalizable" self-interactions are absent and where the gauge is fixed *à la* Feynman. We also looked in some detail at the residual gauge symmetry and the possibility of identifying the physical degrees of freedom in this context. A related question emerged as to whether the 3SUSY algebra would imply rather specific space-time configurations for the partner fields (in contrast with the case of conventional supersymmetry). An unambiguous answer to this question requires the identification of the proper 3SUSY transformation of space-time, for which we only sketched a superspace approach in this paper. The more general question regarding the possibility of an interacting theory would still have to be further investigated. Among the possible directions one could consider coupling the bosonic multiplets to fermionic ones, [albeit highly non conventional kinetic terms for the latter [4]], or more general boson multiplets bi-linear in fermionic fields which are all charged under 3SUSY. One can also consider extended 3SUSY algebras with N copies of the Q generators offering the possibility that the associated automorphism group would induce non-abelian structures. Such a possibility would involve non-abelian self-interacting p-forms which of course require a careful investigation given the strong constraints when $p \geq 2$ (see *e.g* [15]).

Other types of extensions of the Poincaré algebra, namely parasupersymmetric extension have been considered in [16]. A natural question one might address is the relation between these two extensions. Since parasupersymmetry admits interaction terms, this relation (if it exists) could give some indication on the interaction possibilities for 3SUSY.

Finally, a perhaps more promising approach would make use of the fact that 3SUSY has a natural extension in an arbitrary number of space-time dimensions [17], and that the $p-$forms of the bosonic 3SUSY multiplets couple naturally to extended objects of dimension $(p-1)$ ($(p-1)-$branes). The proper transformations of these extended objects are then determined by the 3SUSY transformation of space-time and one can seek for a 3SUSY invariant theory for interacting p-branes.

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